Asymptotic Densities in Statistical Ensembles*

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An expression for the single-eigenvalue (level) density is obtained for a class of ensembles previously considered by the author. The method involves a continuum approximation of a discrete spectrum, and leads to the asymptotic level density as the formal solution of an integral equation. Specific results are calculated for the Jacobi, Hermite, and Laguerre ensembles. These agree with prior calculations for $\beta=1$ and $\beta=2$, but have the feature of containing β as an arbitrary continuously variable parameter. An analogy with a one-dimensional Coulomb gas is used to interpret results and to initiate a search for ensembles with physically realistic densities. A class of ensembles is found which has asymptotic densities whose first and second derivatives are non-negative.

I. INTRODUCTION

SINCE the introduction of the Gaussian ensemble of Hamiltonian matrices by Wigner,[1,2] various investigations of matrix ensembles leading to single-eigenvalue distributions and spacing distributions have ensued.[3] The present work is concerned with an examination of the single-eigenvalue (level) density for a *large class* of statistical ensembles. Specifically, we shall consider ensembles for which the joint-eigenvalue distributions have the form[4]

$$P(x_1,\cdots x_N)=\Omega_{N\beta}^{-1}\Big[\prod_{i=1}^{N} f(x_i)\Big]\prod_{k<l} |x_l-x_k|^{\beta}, \quad (1)$$

where $a\leqslant x_i\leqslant b$ and $\beta>0$. $\Omega_{N\beta}$ is a normalization constant. Our objective is to obtain the asymptotic $(N\to\infty)$ level density formulas corresponding to different choices of $f(x)$, b, and a. This problem has already been investigated to some extent for several ensembles when $\beta=1$,[2,5] and $\beta=2$.[3,4,6–9] For convenience, we shall denote an ensemble represented by (1) by $E_\beta\{f(x)\,|\,(a,b)\}$.

The method employed here was first introduced by Wigner[2] in connection with the Gaussian (Hermite) ensemble of real-symmetric matrices; i.e.,

$$E_1\{\exp(-x^2/\sigma^2)\,|\,(-\infty,\infty)\}.$$

We find that the general approach of Wigner is suitable for treating the entire class of ensembles represented by (1). Specific advantages of the procedure are: (a) It does not depend critically on the value of β, which is allowed to be an arbitrary positive real number; (b) one does not need detailed information about the set of orthogonal polynomials associated with $f(x)$ and (a,b); (c) the dependence of the density on the interval for a *given* $f(x)$ is readily found. Two notable disadvantages should also be stated: (a) The condition of large N is met via a replacement of discrete sums by integrals. This is done *early* in the calculation, and also involves the neglect of certain correlations between the eigenvalues—it is difficult to assess the validity of this procedure *a priori*; (b) a treatment of spacing distributions (where correlations between levels *cannot* be neglected) is evidently not possible using this approach or simple modifications thereof.[10] The former, (a), is not considered serious, because in all cases where comparisons with prior calculations are possible, good agreement is obtained; thus, the approximation scheme appears to be a valid one. The second disadvantage, (b), seems insurmountable. However, indications are that spacing distributions do not depend critically on the ensemble used.[11] *If* this is *universally* true, then the present method can be used in an attempt to find ensembles which yield physically meaningful level densities.

The major result of this paper is that for *finite* intervals (a,b), the level density $\sigma(x)$ can be written down formally in terms of integrations involving $(d/dx)(\ln f(x))$, over the domain (a,b). The integrations can

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[1] E. P. Wigner, Proceedings Gatlinburg Conference on Neutron Physics by Time-of-Flight, ORNL-2309, 67, 1956 (unpublished).
[2] E. P. Wigner, *Proceedings of Fourth Canadian Mathematical Congress* (Toronto University Press, Toronto, Canada, 1959), p. 174.
[3] For a review of matrix ensembles and a detailed bibliography see N. Rosenzweig, in *Statistical Physics* (W. A. Benjamin and Company, Inc., New York, 1963), pp. 91–158. Also, see H. S. Leff, Ph.D. thesis, Research Report 63-23, State University of Iowa, 1963 (unpublished).
[4] H. S. Leff, J. Math. Phys. **5**, 763 (1964).
[5] M. L. Mehta, Nucl. Phys. **18**, 395 (1960); M. L. Mehta and M. Gaudin, *ibid.* **18**, 420 (1960).
[6] E. P. Wigner, in *Statistical Properties of Spectra: Fluctuation*, edited by C. E. Porter (Academic Press Inc., New York, to be published).
[7] P. B. Kahn, C. E. Porter and Y. C. Tang, in Proceedings Eastern Conference in Theoretical Physics, University of North Carolina, October, 1963 (unpublished).
[8] B. V. Bronk, J. Math. Phys. **5**, 215 (1964).
[9] B. V. Bronk (to be published).
[10] A substantial modification of the ideas involved herein (i.e., replacement of a discrete model by a continuum model) has been used to obtain approximate nearest-neighbor spacing distribution formulas for the circular ensembles and the Hermite ensembles. See F. J. Dyson, J. Math. Phys. **3**, 140, 157 (1962).
[11] See the works referred to in footnotes 3, 4, and 10, plus the following. D. Fox and P. B. Kahn, Phys. Rev. **134**, B1151 (1964); F. J. Dyson, J. Math. Phys. **3**, 166 (1962).

be performed exactly for many cases of interest. One can let a or b be very large in order to simulate ensembles over infinite and semi-infinite domains. For the so-called Hermite and Laguerre ensembles (defined in Sec. III) we find that allowable solutions of $\sigma(x)$ can be obtained *only* if b is proportional to $(\beta N)^\gamma$; $\gamma=(\frac{1}{2},1)$ for the (Hermite, Laguerre) cases. For all of the classical ensembles (Jacobi, Hermite, Laguerre) the method yields agreement for those values of β which have already been investigated by other means. All of the present results contain β as an arbitrary positive parameter.

There exists a one-to-one correspondence between the present formalism and certain one-dimensional Coulomb gases. This is used as a tool in the interpretation of the results for the classical ensembles. In this discussion, two unusual situations arise: (i) a temperature-dependent external field is encountered; (ii) the limiting processes do not necessarily coincide with the ordinary thermodynamic limit. The Coulomb gas interpretation is also used to initiate a search for ensembles which have physically realistic level densities. In particular, a class of ensembles is found which has asymptotic level densities whose first and second derivatives are positive semidefinite.

II. GENERAL FORMULATION

This section consists mainly of a review of an approximation scheme due to Wigner.[2] Its application here is to a more general problem than Wigner considered, and we shall depart from his method in one essential way. A discussion of the analogy between the present problem and that involving a certain Coulomb gas is also contained in this section.

It is convenient to work with the logarithm of (1), which effectively converts products to summations:

$$\ln P(x_1,\cdots x_N)=\sum_{i=1}^{N}\ln f(x_i)+\tfrac{1}{2}\sum_{j\neq k}\ln|x_k-x_j|^\beta-\ln\Omega_{N\beta}. \quad (2)$$

For large N, we assume that the summations can be replaced by their corresponding integrations, which allows the introduction of the asymptotic level density $\sigma(x)$:

$$[\ln P]=\int_a^b\sigma(x)\ln f(x)dx+\tfrac{1}{2}\int_a^b\int_a^b\ln|x-y|^\beta\sigma(x)\sigma(y)dxdy-\ln\Omega_{N\beta}. \quad (3)$$

The object denoted symbolically by $[\ln P]$ is a functional of the function $\sigma(x)$. The second term on the right should properly contain a function $\sigma_2(x,y)$, which contains correlations between two levels, one at x and one at y. The replacement of $\sigma_2(x,y)$ by $\sigma(x)\sigma(y)$ neglects these correlations. This is a crucial assumption, but is believed to be a valid one. It is expected that the correlations effectively range over a few mean spacings, and that the $\ln|x-y|^\beta$ factor of the integrand dominates over these in the integral. Another point of importance is the convergence of the integrals in (3). Notice that $\ln P(x_1,\cdots x_N)$ itself will be finite except for "exceptional" configurations where two or more x_j coincide or when $f(x_j)=0$ for one or more values of j. The spirit of the present work is that the actual (observed) configuration of the points $x_1,\cdots x_N$ will be that for which $[\ln P]$ is a maximum. This excludes the possibility of the "exceptional" configurations. In order to insure that (3) exists [which it must, to be a valid representation of (2)] we shall interpret all integrals as principal value integrals. These will be identical with the corresponding Riemann integrals, if they exist. Finally, it is tacitly assumed throughout that the most probable density function (i.e., the one which maximizes $[\ln P]$) is equal to the density function which is obtained by integrating over all but one of the x_j, in $NP(x_1,\cdots x_N)$. The above assumptions can be justified at least partially, *a posteriori* by noting agreement between our results and previous, independent calculations.

The maximization of $[\ln P]$ is to be carried out under the normalization constraint

$$\int_a^b\sigma(x)dx=N. \quad (4)$$

This restriction introduces a Lagrange multiplier K. The condition $\delta\{[\ln P]+K\int_a^b\sigma(x)dx\}=0$ yields the equation

$$\ln f(x)+\beta\int_a^b\sigma(y)\ln|x-y|dy+K=0. \quad (5)$$

K is independent of x. The restriction (4) does not eliminate the possibility of extraneous nonpositive semidefinite solutions. These must be ruled out by appropriate restrictions on a, b, and any free parameters in $f(x)$. In principle, one can solve the integral equation (5), but in practice it is convenient to examine its first derivative. Careful differentiation of (5) yields the following singular integral equation of the first kind:

$$\int_a^b\frac{\sigma(y)}{(y-x)}dy=\beta^{-1}\frac{d}{dx}\ln f(x). \quad (6)$$

At this point, we deviate from Wigner's approach. He was concerned with the Gaussian ensemble, where $b=-a=\infty$.[12] We, however, shall limit ourselves to

[12] For Wigner's particular problem, the solution of (6) appears offhand to entail the inversion of a so-called Hilbert transform. A closer investigation shows that this is not so, and that the inversion integral diverges. For a discussion of Hilbert transforms, see E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals* (Oxford University Press, London, England, 1948), p. 120.

finite values of a and b. While Wigner "guessed" the solution for his particular integral equation, our finite domain restriction allows the use of a rigorous mathematical solution to (6). Following Mikhlin,[13] the general solution of (6) is

$$\sigma(x)=-[\beta\pi^2 R(x,a,b)]^{-1}$$
$$\times\left\{\int_a^b\left[R(y,a,b)\frac{d}{dy}\ln f(y)\right][y-x]^{-1}dy+C_{\beta N}\right\}. \quad (7)$$

Here, $R(x,a,b)=[(x-a)(b-x)]^{1/2}$, and $C_{\beta N}$ is a constant, which can depend parametrically on a, b, β, N, and any parameters which might appear in $f(x)$. The formal evaluation of (7) must be followed by application of the restriction $\sigma(x)\geqslant 0$. Also, in certain cases, one can demand that $\sigma(x)$ satisfies a specific boundary condition at $x=a$ or $x=b$. The constant $C_{\beta N}$ is completely determined by (4).

It is well to point out that there is not a one-to-one correspondence between $\sigma(x)$ and $f(x)$. This is proved by example in Sec. III, where an infinite variety of functions $f(x)$ give rise to the same asymptotic density over a specified interval (a,b). Furthermore, if this density is inserted back into (6) in order to find $f(x)$, the infinite class of $f(x)$ is not recovered. This point will be discussed further in Sec. III.

The connection between $f(x)$ and $\sigma(x)$ can be seen qualitatively by using the Coulomb gas analogy. That is, the present problem is mathematically equivalent to a simple physical model: A gas of N unit charges which (a) exists in a two-dimensional universe, (b) is confined to motion along a one-dimensional line in a *temperature-dependent* external field, and (c) obeys Boltzmann statistics.[14] The total potential energy of the system corresponding to (1) is

$$W=-\sum_{k<j}\ln|x_j-x_k|-\sum_{i=1}^{N}\beta^{-1}\ln f(x_i)+\text{constant}. \quad (8)$$

The second summation corresponds to the temperature-dependent external field W_e. The Boltzmann factor $\exp[-\beta W]$ (omitting the trivial kinetic energy part), is identical to (1). The asymptotic *level* density in our original problem is identical to the asymptotic *particle* density in the corresponding Coulomb gas problem. The word *asymptotic*, as before, refers to the limit $N\to\infty$. This may or may not be equivalent to the ordinary thermodynamic limit: $N\to\infty$, $(b-a)\to\infty$, $N/(b-a)$ = finite. This depends on the precise functional relationships between b and N and a and N.

The force on the ith particle due to the external field is

$$F_e(x_i)=\beta^{-1}(d/dx)\ln f(x_i). \quad (9)$$

By varying $f(x)$, one can evidently "push" the particles so as to obtain qualitatively different particle (level) densities. Because of the fact that a large number N of particles are involved, the effects of different functions $f(x)$ are not completely trivial to analyze. For example, the previously cited situation, where many functions $F_e(x)$ correspond to one $\sigma(x)$, might oppose one's intuition. The interpretation here is that for very large N, certain changes in the external field $F_e(x)$ will not be sufficient to redistribute the particles. We shall rely quite heavily on the Coulomb analogy as a tool in the interpretation of our results.

To close this section, we state a theorem which follows directly from (4)–(7). If the weight function $f(x)$, and corresponding force $F_e(x)$, give rise to an asymptotic density $\sigma(x)$ over $(0,b)$, then the ensemble with force $[-F_e(-x)]$ over $(-b,0)$ has the asymptotic density $\sigma(-x)$. The proof follows from simple change of variable manipulations in (4)–(7).

III. APPLICATION TO THE CLASSICAL ENSEMBLES

We now evaluate (7) for the three so-called classical ensembles, which are based on the weight functions and domains for the Jacobi, Hermite, and Laguerre polynomials. These have already been examined for *specific* values of β.[2–9] The Hermite ensembles, for $\beta=1$, 2, and 4, arise from considerations of invariant ensembles of Hermitian matrices.[3] Specific Jacobi ensembles for $\beta=2$ have arisen in a study of "formally invariant" ensembles of real orthogonal matrices by Dyson.[15] Specific Laguerre ensembles, for $\beta=2$, have also been encountered in Dyson's work with "formally invariant" ensembles—there, in connection with invariant ensembles of anti-Hermitian matrices.[16] A study of Laguerre ensembles has also been pursued by Bronk,[9] in connection with invariant ensembles of negative Hermitian matrices, and also by Kahn, Porter, and Tang.[7] The classical ensembles, specifically, are as follows.

$$\text{Jacobi: } E_\beta\{(1-x)^m(1+x)^n|(-1,1)\},\ m,n>-1. \quad (10)$$

$$\text{Hermite: } E_\beta\{\exp(-x^2/\sigma^2)|(-\infty,\infty)\}. \quad (11)$$

$$\text{Laguerre: } E_\beta\{x^\alpha\exp(-x)|(0,\infty)\},\quad \alpha>-1. \quad (12)$$

Of course, the second two ensembles cannot be treated directly via (7) because infinite limits are involved. As

[13] S. G. Mikhlin, *Integral Equations* (Pergamon Press, Inc., New York, 1957), p. 131.

[14] The Coulomb gas analog was first used by Dyson. See footnote 10.

[15] F. J. Dyson, J. Math. Phys. 3, 1199 (1962). Remarks concerning the real orthogonal matrix ensembles are contained in Ref. 4.

[16] Here, the distribution of the negative of the squares of the eigenvalues are governed by the Laguerre ensembles (for $\beta=2$) of index $\alpha=\pm\frac{1}{2}$, depending on whether the matrices are taken over the real, complex, or quaternion fields, and whether the dimensionality is even or odd.

stated before, we shall replace these by finite, but large, limits. Due to the negative exponential dependences of the weight functions, this "cutoff" is expected to be valid. Specifically, instead of the ensembles (11) and (12), we examine $E_\beta\{\exp(-x^2/\sigma^2)|(-b_H,b_H)\}$ and $E_\beta\{x^\alpha\exp(-x)|(0,b_L)\}$, respectively. The integrals involved in solving (7) for (10)–(12) can be performed quite readily.[17] For the Jacobi ensembles, (10), the result is

$$\sigma_J(x)=(N/\pi)(1-x^2)^{-1/2}, \quad \text{for all} \quad m,n>-1. \qquad (13)$$

For the Hermite and Laguerre ensembles, one finds that solutions to (7), satisfying (4), are

$$\sigma_H(x)=(\sigma^2\beta\pi)^{-1}(b_H^2-x^2)^{-1/2}(\sigma^2N\beta+b_H^2-2x^2) \qquad (14)$$

and

$$\sigma_L(x)=(\beta\pi)^{-1}[x(b_L-x)]^{-1/2} \times(N\beta+b_L/2-x), \quad \text{for all} \quad \alpha>-1, \qquad (15)$$

respectively. Clearly, the subsidiary condition $\sigma(x)\geqslant 0$ is not automatically satisfied by (14) or (15). The application of this restriction leads to the inequalities

$$b_H\leqslant(\sigma^2\beta N)^{1/2}, \qquad (16)$$

$$b_L\leqslant 2\beta N. \qquad (17)$$

One might expect that the best approximations to the true infinite and semi-infinite domain ensembles would be obtained by choosing b_H and b_L to be their maximum allowable values. This, in fact, is the case, the results being

$$\sigma_H(x)=(2/\sigma^2\beta\pi)(\sigma^2\beta N-x^2)^{1/2}, \quad |x|\leqslant(\sigma^2\beta N)^{1/2} \qquad (18)$$

and

$$\sigma_L(x)=(\pi^2\beta^2x)^{-1/2}(2\beta N-x)^{1/2}, \quad 0<x\leqslant 2\beta N, \quad \alpha>-1. \qquad (19)$$

Notice that the choices $b_H=(\sigma^2\beta N)^{1/2}$ and $b_L=(2\beta N)$ are the *only ones* which insure that $\sigma_H(\pm b_H)$ and $\sigma_L(b_L)$ are finite (they are identically zero). This curious situation shows that the approximation scheme is highly sensitive to the choice of intervals. In all applications, one must check to see that the interval is such that $\sigma(x)$ satisfies realistic boundary conditions.

Before discussing these results, we recall that for $\beta=2$, an *exact* expression for $\sigma(x)$ can be written down for arbitrary N. If $p_k(x)$ denotes the orthogonal polynomials which are uniquely defined with respect to $f(x)$ and (a,b), then[4]

$$\sigma(x)=f(x)\sum_{k=0}^{(N-1)}h_k^{-1}p_k^2(x), \qquad (20)$$

where

$$\int_a^b p_k(x)p_j(x)f(x)dx=h_k\delta_{kj}. \qquad (21)$$

While a concise asymptotic formula for $\sigma(x)$ is not necessarily easily obtained from (20), specific features, such as the boundary conditions satisfied by $\sigma(x)$ can be checked with relative ease. We shall find this useful in our analysis.

The Jacobi density, (13), has the notable feature that it is independent of β, m, and n. It also has the property, $\sigma_J(\pm1)=\infty$. However, it is clear from (20) that the latter cannot be so unless m and n are both negative. In fact, if $\{m>0, n>0\}$, then $\{\sigma_J(+1),\sigma_J(-1)\}$ should be identically zero. Also, if $m=n=0$, the true value of $\sigma_J(\pm1)$ is finite, being $\frac{1}{2}N^2$.[4] Equation (13) is, however, in exact agreement with previous findings for $\beta=2$, which are also inaccurate at the end points.[4,7] The present formalism does not yield a precise statement of the domain of validity for the approximation scheme, as do the methods using asymptotic properties of orthogonal polynomials. Here, the domain of validity must be sought via comparisons with (20) and the exploitation of the Coulomb analogy. The latter will now be pursued. The interaction potential energy W_e, for the Jacobi case, is

$$W_e=-\beta^{-1}\sum_{i=1}^{N}[m\ln(1-x_i)+n\ln(1+x_i)]. \qquad (22)$$

For given values of β, m, and n, one can imagine that this is due to two *fixed* charges at $(x=+1, x=-1)$ with magnitudes $(m\beta^{-1},n\beta^{-1})$. If either m or n is negative, there will be an attractive force at $x=+1$ or -1, respectively. This accounts for the infinite densities predicted by (20) for such situations. Similarly, if m or n is positive, there will be a repulsion at $x=+1$ or -1, explaining the zero densities predicted by (20). For very large N ($N\gg|m\beta^{-1}|$, $N\gg|n\beta^{-1}|$), the fixed charges are expected to have little effect on the *gross character* of $\sigma_J(x)$. The cases where m and/or n take on the value zero correspond to the removal of the fixed charges. Here, the natural repulsion between the N charges causes the density to be high, but finite, near the walls. The presence of the fixed charges evidently only affects the behavior of the asymptotic density very near the walls. Since the interval is held fixed as $N\to\infty$, temperature effects are washed out by the Coulomb interactions in the high-density gas; the results are therefore independent of β. The number of particles per unit volume is equal to $\frac{1}{2}N$, which does *not* correspond to the ordinary thermodynamic limit.

The Hermite density, (18), exhibits an explicit dependence on β. This is in agreement with prior calculations for $\beta=1$ and $\beta=2$.[2,5,6] These prior works showed that $\sigma_H(x)\sim 0$ for $|x|>(N\sigma^2)^{1/2}$ when $\beta=1$, and for

[17] One of the variety of ways of doing these integrals is to use integral tables. Particularly recommended is W. Gröbner and N. Hofreiter, *Integraltafel, Unbestimmte Integrale, Erster Teil* (Springer-Verlag, Vienna, Austria, 1957).

$|x|>(2N\sigma^2)^{1/2}$ when $\beta=2$. It is inherent in the present formalism that $\sigma_H(x)=0$ (identically) for $|x|\geqslant(\beta N\sigma^2)^{1/2}$. The strict identity results from the truncation of an infinite domain to one of finite extent, and must be regarded as approximate when the domain is truly infinite. Our formalism does not admit analyses as detailed as Bronk's.[8] In the Coulomb gas analogy,

$$W_e=(\beta\sigma^2)^{-1}\sum_{i=1}^{N}x_i^2. \tag{23}$$

This harmonic oscillator potential opposes the natural tendency of the particles to move toward the walls. The external force is strongest for large values of x, where it dominates over the Coulomb repulsion. As in the Jacobi case, the limiting process does not correspond to the ordinary thermodynamic limit; the mean density here is $N^{1/2}/2\sigma\beta^{1/2}$.

The Laguerre density, (19), has an explicit dependence on β, but is independent of the parameter α. At the origin, it is infinite for all values of α. Certainly, this is a poor approximation for $\alpha>0$, where (20) yields $\sigma_L(0)=0$. The discrepancy is analogous to that involved with the Jacobi ensembles. For $\beta=2$, (19) is in agreement with the results of Bronk,[9] and of Kahn, Porter, and Tang.[7] Their methods are somewhat more precise, yielding specific domains of validity. Bronk's method yields $(\alpha-1)^2/4N<x<4N$. The Coulomb analogy is again useful in the analysis of (19). The potential energy of interaction with the external field is

$$W_e=\beta^{-1}\sum_{i=1}^{N}(x_i-\alpha\ln x_i). \tag{24}$$

For $\alpha=0$, this corresponds to a constant negative electric field, making (19) at least qualitatively reasonable. For $\alpha\neq0$, the second sum in (24) can be thought of as being due to a fixed charge, of strength $\alpha\beta^{-1}$, at the origin. If $\alpha<0$, the Coulomb *attraction* aids the electric field and (19), again, seems plausible. The application of (20) when $\alpha<0$ yields an infinite density at the origin. If $\alpha>0$, the Coulomb *repulsion* opposes the electric field and causes the density very near the origin to be zero. The fact that $N\gg|\alpha\beta^{-1}|$ means that the gross behavior of $\sigma_L(x)$ is virtually unaffected by the fixed charge at the origin. For fixed β, the usual thermodynamic limit *is* satisfied; i.e., the number of particles per unit volume is independent of N, being (2β). By the theorem of Sec. II, the Laguerre ensemble's density (for $\alpha=0$) is simply related to that for the ensemble $E_\beta\{\exp(+x)|(-b,0)\}$. The density for the latter is equal to $\sigma_L(-x)$ for $-2\beta N\leqslant x<0$. This is a monotonically increasing function of x, except for $x\sim0$. The second derivative of $\sigma_L(-x)$ with respect to x is not positive semidefinite.

The Jacobi and Laguerre ensembles are good examples of the fact that a unique one-to-one correspondence does not necessarily exist between $f(x)$ and $\sigma(x)$. Furthermore, as mentioned before, substitution of $\sigma(x)$ into (6) need not recover the entire class of functions $f(x)$ which correspond to the particular function $\sigma(x)$ using (7). Substitution of (13) and (19) into (6) leads to

$$f_J(x)=\text{Const},$$

and

$$f_L(x)=C\exp(-x),$$

respectively; i.e., only the Legendre ensemble and the special Laguerre ensemble for $\alpha=0$ are recovered. The reasons that the more general Jacobi and Laguerre ensembles also give rise to (13) and (19) have already been discussed via the Coulomb gas analogy. The nonunique relationship between $f(x)$ and $\sigma(x)$ can be traced (physically) to the specific asymptotic limits involved.

IV. APPLICATIONS TO OTHER ENSEMBLES

The results of Sec. III indicate that our general formalism is a valid one. It is therefore natural to apply it further. One outstanding question in the statistical theory of energy level spectra is: Can one find matrix ensembles which yield not only the correct spacing distributions, but also a qualitatively correct density of levels? This problem has already been discussed by various people, but no definitive results have arisen.[4,7–9] Both the Jacobi and Laguerre (for negative argument) densities have some desirable properties, but neither simultaneously satisfies the two conditions

$$d\sigma(x)/dx\geqslant0,\quad d^2\sigma(x)/dx^2\geqslant0,\quad\text{for}\quad a<x<b. \tag{25}$$

These are considered to be necessary criteria for what we shall refer to as physically realistic densities.[18]

As a first attempt to find ensembles whose densities satisfy the conditions (25), we explore the weight functions $f(x)=\exp[\pm x^{n+1}/(n+1)]$, $n>0$, over various domains. These are suggested because they correspond to known invariant matrix ensembles involving Trace (H^{n+1}), H being the Hamiltonian matrix. The external force on a particle at the position x for the above $f(x)$ is given by $F_e(x)=\pm(x^n/\beta)$. If the domain is such that $a<0<b$, then there will be a point of zero field in the interval (a,b). When n is an even integer, $F_e(x)$ acts in one direction only, and one expects that $\sigma(x)$ is zero at $x=a$, increases, then dips, and then rises to infinity at $x=b$. The dip is due to the neighborhood about the zero-force field where the particles tend to behave as they do near $x=0$ in the Legendre ensemble. This expectation is borne out by calculations.

If one chooses the interval $(0,b)$ and

$$f(x)=\exp[-x^{n+1}/(n+1)],$$

then one has a situation where the external force is zero at the left wall and is positive elsewhere. This results in

[18] Functions satisfying the second of conditions (25) are members of the set of convex functions. See G. H. Hardy, J. E. Littlewood, and G. Polya, *Inequalities* (Cambridge University Press, London, England, 1934), p. 76.

a density which is infinite at both end points, resembling the Jacobi result (13), except that it is not symmetric about $x=\frac{1}{2}b$.

The interval $(-b,0)$ and weight function

$$f(x)=\exp[(-1)^n x^{n+1}/(n+1)] \tag{26}$$

would appear to be a possibility in yielding densities satisfying (25). Here, the external force $F_e(x)$ satisfies $\beta F_e(x)=(-x)^n$, which is largest for $x=-b$ and smallest for $x=0$, where the natural repulsion raises the density. The densities corresponding to (26) have been calculated for integral values of n from zero to five, and it is found that they follow a definite pattern. Their general form, which we suspect is valid for all positive integers n, is

$$\sigma_{n+1}(x)=(1/\beta\pi)[(x+b_{n+1})/(-x)]^{1/2}\mathcal{L}_n(x), \tag{27}$$

where

$$\mathcal{L}_n(x)=\sum_{j=0}^{n}\{(-1)^{n-j}[(2j)!/2^{2j}(j!)^2]\}b_{n+1}{}^j x^{n-j}, \tag{28}$$

and

$$b_{n+1}{}^{n+1}=\{2^{2n+2}[(n+1)!]^2\beta N/(2n+2)!\}. \tag{29}$$

An examination of (27)–(29) shows that the conditions (25) are *not* satisfied for $n=0,1,2,\cdots$, etc. For $n=0$, the force is constant and is insufficient to cause the second derivative of $\sigma(x)$ to be positive semidefinite. For $n>0$, the maximum force is equal to $\beta^{-1}b_{n+1}{}^n$, but this force diminishes rapidly as x increases, allowing the repulsion to cause a violation of one or both of the conditions (25).

The approach of *guessing* weight functions in order to obtain a density satisfying (25) is not a very efficient one. A more direct approach is to choose specific functions $\sigma(x)$ which satisfy (25), and find $f(x)$ via (6). Toward this end, we investigate the case

$$\sigma_m(x)=\beta^{-1}x^m,\quad m=0,1,\text{ etc. }\cdots,$$
$$a=0\quad,\quad b^{m+1}=(m+1)\beta N. \tag{30}$$

The evaluation of (6) for (30) yields

$$F_e(x)=\beta^{-1}\sum_{\nu=0}^{m-1}[b^{m-\nu}/(m-\nu)][1-\delta_{m0}]x^\nu$$
$$+\beta^{-1}x^m\ln[(b-x)/x]. \tag{31}$$

The corresponding weight function, within a multiplicative constant, is

$$f(x)=x^{-(x^{m+1}/m+1)}(b-x)^{Q(b-x)}$$
$$\times\exp[R(x)+S(b-x)]. \tag{32}$$

$Q(b-x)$, $R(x)$, and $S(b-x)$ are polynomials of degree $m+1$ in the shown arguments. For each, the coefficient of the zeroth power is zero. Equations (31) and (32) have the following features: (i) $f(0)=b^{-\gamma_1 b^{m+1}}\exp(\gamma_2 b^{m+1})$; $\beta F_e(0)=b^m/m$. (ii) $f(b)=b^{-b^{m+1}/m+1}\exp(\gamma_3 b^{m+1})$; $F_e(b)=-\infty$. γ_1, γ_2, and γ_3 are positive semidefinite functions of m. (iii) $F_e(x)$ is large and positive over enough of the domain to require a negatively infinite force at $x=b$ in order to insure a finite density at that point. Thus, we have a class of ensembles which gives rise to densities satisfying (25). They are relatively complicated, and the insertion of (32) back into (7) leads to integrals which are nontrivial (if at all possible) to evaluate. It is far from obvious which invariant matrix ensembles correspond to joint eigenvalue distributions given by (1) and (32). Finally, it is not known whether the form (32) is *necessary* in order for (25) to be fulfilled, but only that it is *sufficient*.

The densities (30) have been used because the integrals could be performed quite readily. A natural extension would be to insert actual level density formulas into (6). Such formulas have been investigated for nuclei by many people.[19] They invariably involve a positive exponential dependence on the energy, with a typical form being

$$\sigma(x)\sim x^p\exp(qx-r)^{1/2},\quad q>0. \tag{33}$$

Insertion of (33) into (7) leads to integrals of prohibitive difficulty. A rough argument, based on the fact that (33) can be well approximated over a finite domain by a polynomial leads us to the conclusion that a linear combination of "external forces" (31) can be chosen which will yield a density function very much like (33). A simple example is a small x expansion of $F_e(x)$ for $\sigma(x)=\beta^{-1}\exp(\lambda x)$.[20] For this case, the small x behavior is dominated by $-\beta^{-1}\exp(\lambda x)\ln x$ which is what would be predicted using the above superposition argument.

One can also employ the present scheme to gain insight into ensembles which have densities with a gap property; i.e., which are identically zero over some finite subinterval of (a,b). The investigation of such ensembles has already been initiated by other means.[7,21] A straightforward manipulation of the foregoing ideas can be used to obtain a class of ensembles which display "gaps" in their density functions.

V. DISCUSSION

We have seen that expressions (6) and (7) can be directly applied to a number of cases of interest. In the statistical theory of spectra, β is only allowed to take on the values 1, 2, or 4.[3] However, β clearly determines the degree of repulsion between the levels (particles), and values of β other than 1, 2, or 4 might be meaningful in connection with the problem of approximate quantum

[19] See, for example, T. D. Newton, Can. J. Phys. **34**, 804 (1956); N. Rosenzweig, Phys. Rev. **105**, 950 (1957); **108**, 817 (1957); T. Ericson, Phil. Mag. Suppl. **9**, 425 (1960).

[20] Here, the "exponential integrals" Ei(x), which cannot be evaluated in closed form, are encountered. See Jahnke and F. Emde, *Tables of Functions* (G. E. Stechert and Company, New York, 1938), p. 1.

[21] C. E. Porter, J. Math. Phys. 4, 1039 (1963).

numbers,[22] or perhaps in the treatment of superposed sequences.[23] In both of these cases, the repulsion effect is reduced from its magnitude in a set of energy levels having a given, precise symmetry character. Such studies would evidently be useful only if local properties, such as spacing distributions, could be found as a function of β. This is not possible with the present formalism.

The Coulomb analogy allows us to draw some conclusions about certain ensembles without making any calculations at all. For example, Bronk[9] has suggested the study of $E_\beta\{x^{1/2}\exp(-\lambda x^2)\,|\,(-\infty,0)\}$ in connection with finding a convex density [satisfying (25)]. The external potential energy of interaction for this case is

$$W_e=-\tfrac{1}{2}\beta^{-1}\sum_{i=1}^{N}\ln x_i+\lambda\beta^{-1}\sum_{i=1}^{N}x_i^2. \qquad (34)$$

The second set of terms would arise in

$$E_\beta\{\exp(-\lambda x^2)\,|\,(-\infty,0)\},$$

and the first set of terms correspond to an interaction with a fixed charge of magnitude $\frac{1}{2}\beta^{-1}$, located at the origin. From our experiences in Sec. III, we do not expect the fixed charge to change the density, except in the immediate neighborhood of the origin. We may conclude that the two above ensembles have the *same* asymptotic density. This density, for $\lambda=\frac{1}{2}$, is given by (27)–(29) for $n=1$, and does *not* satisfy the conditions (25).

In closing, we recall that our formalism demands a finite interval (a,b) and we can only *approach* infinite and semi-infinite intervals. However, it should be remarked that it is this very fact which allows us to use the Coulomb analogy, which would be senseless in an infinite domain. There is no reason (other than convenience) to favor symmetric domains over nonsymmetric domains or vice versa. The only factors which determine the asymptotic density are (i) the values of $f(x)$ over (a,b), and (ii) the constraints imposed on the "volume" $(b-a)$. For the Hermite and Laguerre cases the level densities behave, very roughly, like the corresponding weight functions $f(x)$ themselves. The Jacobi density does not share this feature and here, temperature effects are washed out. Indications are that for fixed intervals, the strong repulsion dominates, resulting in a behavior similar to (13). For intervals which are proportional to some positive power of N, the density resembles (crudely) $f(x)$. If $\beta=0$, the integral equation approach is no longer meaningful, and the densities for such cases are exactly proportional to $f(x)$. For such cases, which correspond to infinite temperatures, the particles (levels) are statistically independent.

Note added in proof. A further verification that Eq. (18) is correct is its agreement with $\sigma_H(0)$ for $\beta=4$, given by M. L. Mehta and F. J. Dyson, J. Math. Phys. 4, 713 (1963).

[22] N. Rosenzweig and C. E. Porter, Phys. Rev. **120**, 1698 (1960); F. J. Dyson, J. Math. Phys. 3, 1191 (1962); N. Rosenzweig, Bull. Am. Phys. Soc. 8, 263 (1963).

[23] H. S. Leff, J. Math. Phys. 5, 756 (1964); Bull. Am. Phys. Soc. 8, 31 (1963). Also see the first of Refs. 22 and the second of Refs. 3.